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LETTER TO THE EDITOR

# On the evaluation of the class operator for the rotation group

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**Abstract.** A simple derivation is given of the explicit value of the class operator for the rotation group SO(3).

In a recent paper by Hongyi and Yong [1] it was shown that the class operator

$$\bar{C}(\psi)=\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\,\exp[\mathrm{i}\psi(J_x\sin\theta\cos\phi+J_y\sin\theta\sin\phi+J_z\cos\theta)] \qquad (1)$$

where $J_x, J_y, J_z$ are infinitesimal generators of the rotation group SO(3) (or SU(2)) has explicit value

$$\sin(\tfrac{1}{2}\psi t)/t\sin(\tfrac{1}{2}\psi) \qquad (2)$$

where $t$ is related to the Casimir element $J^2=J_x^2+J_y^2+J_z^2$ by $t^2-1=4J^2$.

We should interpret this in the sense that on expansion of (1) and (2) as power series in $\psi$, and in the case of (1) performing the integration over the angular variables, both become infinite series in the generators $J_x, J_y, J_z$. In each case, the coefficient of $\psi^k$ is a finite polynomial of degree at most $k$ in the generators, $k=0,1,2,\ldots$, and therefore belongs to the enveloping algebra of the Lie algebra with commutation relations

$$[J_a, J_b]=\mathrm{i}\varepsilon_{abc}J_c \qquad a, b, c=(x, y, z). \qquad (3)$$

The equivalence of (1) and (2) simply means that their respective coefficient polynomials coincide at all degrees in $\psi$ as elements of the enveloping algebra.

We remark that the potentially troublesome factor $t$ in the denominator of (2) cancels when the numerator is expanded as a series in $\psi$. Furthermore, the expansion of (2) only involves $t^2$ and therefore is a function of $J^2$.

The equality between (1) and (2) was proved directly by Hongyi and Yong using an elegant technique based on the Schwinger representation, coherent states and normal ordering. In this letter we give an alternative derivation exploiting explicit matrix representations of SU(2) in a very simple way. The analysis can be extended to other Lie groups, which is not so obviously true of the method given in [1].

We first observe that the operator (1) commutes with all elements of the group. This is because the integration is performed over the group manifold of SU(2) factored by its maximal torus U(1), with respect to the standard invariant measure on this coset space. It follows that (1) commutes with the infinitesimal generators and therefore so do the coefficient polynomials of each power of $\psi$ in the expansion. Thus each coefficient polynomial lies in the centre of the enveloping algebra and must be a finite polynomial in $J^2$.

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This is encouraging since we have already observed that (2) only involves the generators through the combination $J^2$.

The next step is to evaluate (1) and (2) as matrices when the generators are replaced by their matrix representatives in the irreducible representation of dimension $2j+1$. In this spin-$j$ representation, $J^2$ is the scalar matrix with diagonal entries $j(j+1)$. Then both (1) and (2) become scalar matrices and are clearly determined completely by their traces. Evidently, $t$ becomes the scalar matrix with diagonal entries $2j+1$. Thus

$$\begin{aligned}\mathrm{Tr}_j \sin(\tfrac{1}{2}\psi t)/t \sin(\tfrac{1}{2}\psi) &= \frac{(2j+1)\sin[(j+\frac{1}{2})\psi]}{(2j+1)\sin(\frac{1}{2}\psi)} \\ &= \frac{\sin[(j+\frac{1}{2})\psi]}{\sin(\frac{1}{2}\psi)} \end{aligned} \tag{4}$$

a familiar expression as an irreducible character, where $\mathrm{Tr}_j$ means the trace is evaluated in the spin-$j$ representation.

Now consider $\mathrm{Tr}_j\, \bar{C}(\psi)$. The trace can be taken inside the integral and applied to the exponential. The latter is the matrix representing a group element of a rotation through an angle $\psi$ $(0 \leqslant \psi \leqslant 4\pi)$. This is matrix equivalent to a matrix representing a rotation through the same angle in the maximal terms U(1), i.e. the spin-$j$ representation this matrix

$$\mathrm{diag}(\mathrm{e}^{\mathrm{i}k\psi}, k = j, i-1, \ldots, -j). \tag{5}$$

Since trace is invariant under matrix equivalence, the trace of the exponential in (1) is $\Sigma_{k=-j}^{j} \mathrm{e}^{\mathrm{i}k\psi}$, which is obviously independent of $\theta$ and $\phi$, and is known to coincide with (4). Performing the angular integration which simply removes the factor $4\pi$, we see that (1) and (2) coincide at the spin-$j$ representation for all $j = 0, \frac{1}{2}, 1, \ldots$.

We now argue that (1) and (2) coincide as formal series in the enveloping algebra. The difference between (1) and (2) can be expressed in the form

$$\sum_{k=1}^{\infty} f_k(J^2)\psi^{2k} \tag{6}$$

where $f_k$ is a polynomial of degree $\leqslant k$ in the single operator $J^2$. Treating $\psi$ as an indeterminate, we have shown in the paragraphs above that $f_k(j(j+1)) = 0$ for each integer $k \geqslant 1$ and for all $j(j+1), j = 0, \frac{1}{2}, 1, \ldots$. But a polynomial of finite degree cannot vanish at an infinite number of distinct points unless it is identically zero. Thus (6) is identically zero and the expressions (1) and (2) coincide identically.

We conclude with some remarks giving a wider perspective on the class operator and its evaluation for other groups.

One might first encounter the equivalent of (1) in the representation theory of finite groups. In this context, what stands in analogy to the class operator is the sum of elements. More precisely, there is a function from the set of conjugacy classes of the group to its group algebra such that the value of the function at a given class is the sum of the group elements in that class. These class sums are central elements and indeed form a linear basis for the centre of the group algebra. Evaluated on an irreducible matrix representation, each class sum is represented by a scalar matrix in which the diagonal value is a character value. This line of argument, going back at least to Burnside [2], forms the basis for a method, albeit of limited practicality, for computing character titles from class theoretic data and involving matrix diagonalisation. See also Chen and Birman [3] and Backhouse [4].

In the case of compact Lie groups the set of conjugacy classes is in one-to-one correspondence with the points of the maximal torus $U(1) \times U(1) \times \ldots \times U(1)$, $r$ times, where $r$ is the rank of the group. The class operator involves invariant integration over the coset space of the group factored by its maximal torus. We thus have a function $C(\psi_1, \psi_2, \ldots, \psi_r)$, $\psi_i \in U(1)$, $1 \leq i \leq r$, whose values are central formal power series in the enveloping algebra of the group. For a semisimple Lie group the centre of the enveloping algebra is generated by $r$ independent Casimir elements, these being finite polynomials in the infinitesimal generators of the group.

Suppose the irreducible representations of the group are labelled by elements $j \equiv (j_1, j_2, \ldots, j_r)$ from some index set. Let $d_j$ denote the dimension of the $j$th irreducible representation and $\chi_j(\psi_1, \psi_2, \ldots, \psi_r)$ its character, considered as a complex-valued function on the maximal torus. The latter is given as a quotient of trigonometric functions by the Weyl character formula—expression (4) is of course the simplest example of this.

Now, if $C$ is evaluated at the $j$th representation, it becomes a scalar matrix whose diagonal value is some complex function $C_j(\psi_1, \psi_2, \ldots, \psi_r)$. By the method of taking traces, used in § 2, we see that

$$C_j = \chi_j / d_j. \qquad (7)$$

The next step in computing $C$ explicitly involves the Casimirs. At an irreducible representation the Casimirs assume scalar values which are functions of the $j$ parameters. Furthermore, it is in principle possible to invert these functions, thereby expressing the $j$ parameters in terms of the values of the Casimir elements. Equation (7) can now be re-expressed in terms of the latter. Finally $C$ is obtained from $C_j$ simply by replacing Casimir values by the actual Casimirs themselves.

## References

[1] Hongyi F and Yong R 1988 *J. Phys. A: Math. Gen.* **21** 1971-6
[2] Burnside W 1955 *Theory of Groups of Finite Order* (New York: Dover)
[3] Chen L C and Birman J L 1971 *J. Math. Phys.* **12** 2454-61
[4] Backhouse N B 1983 *Physica* **70** 205-19